Tachyon Condensation on Fuzzy Sphere and Noncommutative Solitons

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Abstract

We study a brane-antibrane system and a non-BPS D-brane in $SU(2)$ WZW model. We first discuss the tachyon condensation using the vertex operator formalism and find the generation of codimension two D-branes after the condensation. Our result is consistent with the recent interpretation that a D2-brane is a bound state of D0-branes. Then we investigate the world-volume effective theory on a non-BPS D-brane. It becomes a field theory on the "fuzzy sphere" when the level is sent to infinity. The most interesting feature is that there exist the noncommutative tachyonic solitons and we can identify them with D0-branes. We also discuss the brane-antibrane system from the world-volume point of view and comment on the relation to the noncommutative version of the index theorem.

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1 Introduction

The study of tachyon condensation [1, 2] in open string theory has been important to know the dynamics of non-BPS states in superstring theory. Up to now active investigations have been made in this area [3]. In particular Sen and Witten pointed out that a tachyon configuration on a non-supersymmetric brane system generates a lower dimensional brane system if the configuration has a non-trivial topology or K-theory charge [4, 5, 6]. One way to verify this conjecture is to use the description of the boundary conformal field theory [4, 7, 8, 9, 10] and their boundary state descriptions were given in [11, 12].

Non-BPS solitonic objects in superstring theory are classified into two types. One is a brane-antibrane system [1, 2, 4, 7] and the other is a non-BPS D-brane [13, 4, 7, 14, 15]. Both systems include open strings which have the opposite GSO-projection and the tachyonic instability occurs in both cases.

So far most of the examples in this subject have been studied only in flat space or various orbifolds. As a next step one would like to discuss the dynamics of non-BPS states which consist of curved D-branes in more general backgrounds such as group manifolds. Here we will consider D-branes in $SU(2)$ WZW model. Such a background indeed appears in the near horizon limit of NS5-branes [16]. A D-brane in $SU(2)$ WZW model is described by a boundary state which satisfies the Cardy's condition [17]. It describes a D2-brane wrapping on a conjugacy class of $SU(2)$ [18]. Moreover the low energy effective theory on the brane turns out to be a noncommutative theory because of the background H-flux [19, 20]. In particular the algebraic structure of this theory is identified with that of a fuzzy sphere [21, 22] in the infinite level limit.

The purpose of this paper is to study the tachyon condensation on a non-BPS D-brane or a pair of brane-antibrane in $SU(2)$ WZW model from two different points of view. The first approach is to investigate the vertex operators which correspond to various tachyonic modes. In the discussion below we will determine what is generated after the tachyon condensation when we apply the Sen's conjecture. Interestingly the results are consistent with the bound state interpretation argued in [23, 24]. The second is to study the low energy effective action of the tachyon field. In this approach it is crucial to take the infinite level limit, which corresponds to the infinite B-field limit. The notable point is that the effective action is dramatically simplified, which reflects the fact that the world-volume of a D-brane can be regarded as a fuzzy sphere. Consequently this leads to the interesting observation that an excitation of the tachyon field on the non-BPS D-brane is very much like the noncommutative soliton [25]. This can be seen as a "noncommutative tachyon" [26, 27, 28] on the fuzzy sphere. Indeed we can see that this soliton approaches the known noncommutative soliton in a flat space in the large world-volume limit.

The paper is organized as follows. In section 2 we first review the boundary state description of D-branes in $SU(2)$ WZW model. After that we investigate the tachyon vertex operators and discuss the generation of codimension two D-branes on the world-volume of a pair of brane-antibrane. In section 3 we discuss excitations of the tachyon field in the effective theory on a non-BPS D-brane and show that these can be identified with "noncommutative solitons" if we take the limit of level infinity. We also consider the brane-antibrane system and mention the relation between the tachyon condensation and the index theorem of a noncommutative algebra.

2 Open strings in $SU(2)$ WZW model and tachyon condensation

In this section first we review the description of open strings between D-branes in $SU(2)$ WZW model and then discuss tachyonic modes on brane-antibrane systems and non-BPS D-branes. $SU(2)$ WZW model with level k appears naturally in the near-horizon geometry of N parallel NS 5-branes [16] when one is interested in the critical superstring theory; the relation between the level k and N is given by $N = k + 2$. The total geometry is

$$\mathbf{R}^{5,1} \times \mathbf{R}_\phi \times \mathbf{S}^3, \tag{2.1}$$

where $\mathbf{R}_\phi$ is the radial direction.

This radial direction is described by the linear dilaton theory. The boundary condition along this direction is Neumann or Dirichlet [29]. Under the Neumann boundary condition, the worldsheet fermions are the same as in the flat space. Therefore D-branes preserve some supersymmetries. On the other hand, the Dirichlet boundary condition seems to break supersymmetry and the explicit Dirichlet boundary state in supersymmetric theories has not been known yet.

Throughout this paper we will omit the $\mathbf{R}^{5,1} \times \mathbf{R}_\phi$ geometry and study explicitly only the $\mathbf{S^3}$ geometry. We also assume that the coupling constant in the string theory is so small that we can trust the tree level analysis.

2.1 D-branes in $SU(2)$ WZW model

In rational conformal field theories which have the diagonal modular invariance, one can construct the boundary states of D-branes by the Cardy's prescription [17] in general. The result is

$$|p\rangle = \sum_{a=1}^{N-1} \frac{S_{pa}}{\sqrt{S_{1a}}} |a\rangle\rangle \qquad (1 \le p \le N-1), \tag{2.2}$$

where $|a\rangle\rangle$ is called the Ishibashi state [30] which corresponds to the highest weight state of spin $\frac{a-1}{2}$; the modular transformation matrix is denoted by

$$S_{pa} = \sqrt{\frac{2}{N}} \sin(\frac{pa\pi}{N}). \tag{2.3}$$

The geometrical interpretation of these D-branes was first given in [18] and further discussed[4] in [19, 20, 23, 24, 33, 34]. $SU(2)$ WZW model can be interpreted as a sigma model of which target space is $SU(2) \simeq \mathbf{S^3}$ with $N = k+2$ units of H-flux. The shift of level by two is due to the contribution of fermionic sector and the radius of $\mathbf{S^3}$ is $\sqrt{N\alpha'}$. Let us define the coordinates of $\mathbf{S}^3$ as

$$(\cos\psi, \sin\psi\cos\phi\sin\theta, \sin\psi\sin\phi\sin\theta, \sin\psi\cos\theta) \in \mathbf{R}^4. \tag{2.4}$$

Then a D-brane $|p\rangle$ can be identified with the one wrapping on the p-th conjugacy class of $SU(2)$, which corresponds to $\mathbf{S^2} \subset \mathbf{S^3}$ with $\psi = \frac{p\pi}{N}$ [18]. A D-brane corresponding to $p = 1$ or $p = N-1$ is equivalent to a pure D0-brane and the other branes can be regarded as D2-branes. Even though these D2-branes wrap on the topologically trivial 2-cycles, they are stabilized by the combined effect of the H-flux and the gauge flux

$$F = -\frac{p}{2}\sin\theta d\theta d\phi \tag{2.5}$$

on the world-volume [23]. This shows that such a D-brane can be interpreted as a bound state of p D0-branes [23] and it was pointed out in [20] that a stuck of D0-branes can condense into a single D2-brane.

The open string spectrum between a pair of D-branes corresponding to $|p\rangle$ and $|q\rangle$ is represented as the following cylinder amplitude

$$\begin{aligned} Z_{pq}(q) &= \langle p|\tilde{q}^{L_0-\frac{c}{24}}|q\rangle = \sum_j n^j_{pq}\chi_j(q), \\ \chi_j(q) &= \mathrm{Tr}_j(q^{L_0-\frac{c}{24}}), \end{aligned} \tag{2.6}$$

where $\tilde{q} = e^{-2\frac{\pi}{t}}$ and $q = e^{-2\pi t}$ are closed string and open string moduli of the cylinder, respectively. The character $\chi_j(q)$ means the partition function of the open string which belongs to spin j sector. A notable point is the appearance of the fusion coefficient $n^j_{pq} \in \mathbf{Z}$ as discussed in [17] . In the case of $SU(2)$ WZW model, n^j_{pq} is given as follows

$$\begin{aligned} n^j_{pq} &= 1 \text{ if } \frac{|p-q|}{2} \leq j \leq \min\{\frac{p+q}{2} - 1,\ N-1-\frac{p+q}{2}\}, \\ n^j_{pq} &= 0 \text{ elsewhere }. \end{aligned} \tag{2.7}$$

[4]For the earlier discussions of this subject, see [31, 32].

Then the lightest modes or equally the zero modes of the open strings can be represented as the following vertex operators

$$V_{j,m} \ : \ -j \leq m \leq j \ , \quad \frac{|p-q|}{2} \leq j \leq \min\{\frac{p+q}{2}-1, \ N-1-\frac{p+q}{2}\}. \tag{2.8}$$

Their conformal dimensions are given by

$$\Delta_{j,m} = \frac{j(j+1)}{N}. \tag{2.9}$$

Now we turn to the ten dimensional string theory including the non-compact directions and assume the Neumann boundary condition for $\mathbf{R}_\phi$ direction. Then we obtain the mass spectrum for the above vertex operators as

$$\alpha' m^2 = -\frac{1}{2} + \frac{j(j+1)}{N} + \frac{Q^2}{8}, \tag{2.10}$$

where $Q = \sqrt{\frac{2}{N}}$ is the background charge for the linear dilaton sector. Here we used normalizable states, of which spectrum has the lower bound $\frac{Q^2}{8}$ and its description by using $SL(2,\mathbf{R})/U(1)$ model was discussed in [34]. To see the lowest mass states, we assumed that there is no momentum along any coordinates except $\mathbf{S^3}$. From this we can see that the term $\frac{Q^2}{8} = \frac{1}{4N}$ does not change the qualitative behavior of tachyon fields. In particular if we take $N \to \infty$ limit, then we can neglect this. Though the explicit form of the Dirichlet boundary state has not been known in the supersymmetric linear dilaton theory, we assume that the similar behavior will occur.

2.2 Tachyon condensation in $SU(2)$ WZW model

Let us turn to the tachyon condensation on a pair of brane-antibrane or on a non-BPS D-brane in $SU(2)$ WZW model. As explained in the previous subsection, only D2-branes ($|2\rangle, |3\rangle, \cdots, |N-2\rangle$) and D0-branes ($|1\rangle, |N-1\rangle$) are allowed in this model whether the system is BPS or not. Here we assume N is finite and therefore the conformal dimensions of the vertex operators with different spins j are not the same.

First we consider a brane-antibrane system where both branes correspond to the boundary state $|p\rangle$[5]. In order to distinguish them, we denote the brane by $|p\rangle$ and the antibrane by $|\bar{p}\rangle$. We can assume $1 \leq p \leq \frac{N}{2}$ without loss of generality. Then the open strings between D-brane and anti D-brane have the opposite GSO-projection and the "tachyonic mode" appears once for each vertex operator $V_{j,m}$ $(j = 0, 1, \cdots, p-1)$. If we

[5]Notice that an antibrane have the same bosonic part of the boundary state as a brane. However if one take the fermionic part into consideration, then each has a different sign of the RR-sector.

assume $p < \sqrt{\frac{N}{2}}$, then all of these modes are really tachyonic ($m^2 < 0$). Even if $p > \sqrt{\frac{N}{2}}$, these operators do not include any oscillator excitations and can be considered to belong to the sector of tachyon field. Then let us give an interpretation of these modes from the viewpoint of the world-volume theory. The world-volume of the brane-antibrane system is $\mathbf{S^2}$ and its radius is $\sqrt{N\alpha'} \sin \frac{\pi p}{N}$. The low energy effective theory consists of the (complex) tachyon field and the massless fields (a gauge field and a scalar field). The crucial observation is that the tachyonic mode $V_{j,m}$ in string theory corresponds to the tachyon field on the world-volume which is proportional to the spherical harmonics $Y_{j,m}(\theta, \phi)$. This interpretation is very natural as implied in [33, 20] and an elementary explanation is as follows. If one restricts the $SU(2)$ currents in the WZW model to zero modes (θ, ϕ) on the sphere and quantizes the modes, then one obtains the conventional angular momentum operator in quantum mechanics. For example, the mode $V_{j,j}$ represents the tachyon field

$$T(\theta, \phi) = T_0 \, e^{ij\phi} (\sin\theta)^j, \tag{2.11}$$

where T_0 is a constant. Note that this field has two nodes $T(\theta, \phi) = 0$ at $\theta = 0, \pi$ and they correspond to the vortex line configuration of winding number $j, -j$, respectively.

In general, n codimension two D-branes are generated after the tachyon condensation of winding number n vortex line [4, 5, 9, 12]. Thus we can conclude that the condensation of a tachyonic mode $V_{j,j}$ produces j D0-branes at $\theta = 0$ and j anti D0-branes at $\theta = \pi$. This result is consistent with the allowed values $j = 0, 1, \cdots, p-1$, because the original (anti) D2-brane $|p\rangle$ ($|\bar{p}\rangle$) can be thought as a bound state of p (anti) D0-branes as we mentioned above. For example, the condensation of $j = p-1$ mode means the annihilation of only one D0 and anti D0-brane; the constant mode $j = 0$ corresponds to the annihilation of all branes and the system will eventually go down to the vacuum. Consequently we can say that this gives another evidence of the bound state interpretation discussed in [23, 20]. The similar argument is applicable if $m = -j$.

Next let us discuss the mode $V_{j,m}$ ($|m| \neq j$). Naively one may think that because the corresponding tachyon field proportional to $Y_{j,m}$ includes one dimensional nodes $Y_{j,m} = 0$, its condensation will produce codimension one D-branes according to the Sen's conjecture. However it is easy to see that one can not construct boundary states of such D-branes in our model. We argue that the solution to this puzzle is the following. If we condense the tachyonic mode $V_{j,m}$(or equally give the relevant perturbation in the sense of [35]), then the OPEs of more than three insertions at the worldsheet boundary include the mode $V_{m,m}$ because of the conservation of the angular momentum. Since this mode is more relevant than $V_{j,m}$, the condensation of $V_{j,m}$ results in the previous case and the generations of codimension one D-branes do not really occur.

Then we turn to the case where a D-brane corresponds to $|p\rangle$ and an anti D-brane to $|\bar{q}\rangle$ $(p \neq q)$. Without losing generality, we can assume $p > q$ and $\frac{p+q-2}{2} \leq N - 1 - \frac{p+q}{2}$. Then the open string spectrum of zero modes is given by $V_{j,m}$ $(j = \frac{p-q}{2}, \frac{p-q+2}{2}, \cdots, \frac{p+q-2}{2})$. At first sight one may be in trouble with the fact that if $p-q$ is odd, then one will get the "double valued spherical harmonics" and fractional D0-branes seem to be generated after the tachyon condensation. To resolve this one should note the crucial point that the open string between these branes is affected by the gauge flux $F^{(p)}$, $F^{(q)}$ on each world-volume. In other words the tachyonic field $T(\theta, \phi)$ in this case is not a function on $\mathbf{S}^2$ but a section of a line bundle which couples to the difference of the gauge fields

$$A^{(p)} - A^{(q)} = \frac{p-q}{2} \cos\theta d\phi. \tag{2.12}$$

Therefore the tachyon field is affected by the gauge holonomy as follows

$$T(\theta, \phi) = T_0 \cdot Y_{j,m} \cdot e^{i\int_0^\phi d\phi (A_\phi^{(p)} - A_\phi^{(q)})} = T_0 \cdot Y_{j,m} e^{i\frac{p-q}{2} cos\theta \cdot \phi}, \tag{2.13}$$

where T_0 is a constant again. Thus we get $(m + \frac{p-q}{2})$ D0-branes at $\theta = 0$ and $(m - \frac{p-q}{2})$ anti D0-branes at $\theta = \pi$ without any emergence of the fractional D0-branes. For instance if we consider[6] the lowest mode $j = m = \frac{p-q}{2}$, then we get $(p-q)$ D0-branes only at $\theta = 0$. In the case of the highest mode $j = m = \frac{p+q}{2} - 1$ the tachyon condensation generates $(p-1)$ D0-branes at $\theta = 0$ and $(q-1)$ anti D0-branes at $\theta = \pi$ and this corresponds to one pair annihilation. These observations are again consistent with the interpretation that a D2-brane of type $|p\rangle$ is a bound state of p D0-branes.

Finally let us turn to the tachyon condensation on a non-BPS D2-brane in superstring or a D2-brane in bosonic string. In these cases the tachyon field is a real scalar field and only the linear combinations such as $V_{j,m} + V_{j,-m}$, $i(V_{j,m} - V_{j,-m})$ are allowed. Therefore we cannot gain the vortex line configurations and it is hard to tell what is generated after the condensation. We will investigate this issue in the next section from a different point of view.

3 Tachyon condensation on fuzzy sphere and noncommutative solitons

In this section we investigate the tachyon condensation from the viewpoint of the world-volume effective theory. Here we take the limit $N \to \infty$[7] and in this limit the algebra of

[6]Here again we have only to consider the cases $m = j, -j$ as in the previous discussion.

[7]Roughly speaking, this corresponds to the large B-field limit which is taken in [26, 27, 28], where D-branes in a flat space are considered. Quite recently, the construction of the noncommutative soliton without taking this limit is discussed in [36, 37].

vertex operators in $SU(2)$ WZW model is equivalent to that of the fuzzy sphere [21, 22] as shown in [19]. A fuzzy sphere is defined by identifying the following $SU(2)$ algebra with its coordinates

$$[X^i, X^j] = i\frac{2}{\sqrt{p^2-1}}\epsilon_{ijk}X^k, \quad (X^1)^2 + (X^2)^2 + (X^3)^2 = 1, \tag{3.1}$$

where X^i $(i = 1, 2, 3)$ are $p \times p$ matrices. The integer p labels the algebra of fuzzy sphere and it can be identified with the label p of D2-brane.

In the discussion below, we will study the tachyon condensation on a non-BPS D-brane, which generates codimension two non-BPS D-branes and then investigate the brane-antibrane systems.

The effective action which describes the gauge theory on a (BPS) D2-brane was constructed in [20] and its interesting structure was explicitly shown. Now we would like to consider the world-volume theory on a non-BPS D2-brane corresponding to $|p\rangle$. This includes a real tachyon field T which is described by a (quantum mechanical) $p \times p$ hermitian matrix. In the large radius limit $(p \to \infty)$ the world-volume approaches a flat space and the dynamics is described by a noncommutative field theory. Its action is given by

$$\begin{gathered} S = \frac{1}{G_s}\int dt(dx)^2\sqrt{G}[G^{\mu\nu}\partial_\mu T\partial_\nu T + V(*T)], \\ V(*T) = G_s T_{D2} - \frac{1}{2\alpha'}T*T + \cdots. \end{gathered} \tag{3.2}$$

Here we used the open string metric $G^{\mu\nu}$, the effective open string coupling G_s and the star product $A * B(x) = e^{\frac{1}{2}\Theta^{ij}\partial_i\partial'_j}A(x)B(x')|_{x'=x}$ as in the conventional noncommutative field theory description [38]. Also we defined T_{Dp} as a Dp-brane tension.

When we take finite p (i.e. the world-volume is a fuzzy sphere) we obtain the "regularized version" of the above action as follows

$$S \;=\; \frac{2\pi\Theta}{G_s}\int dt\sqrt{G}\mathrm{Tr}\left[f(T)\sum_{i=1,2,3}T[X^i,[X^i,T]] + V(T)\right], \tag{3.3}$$

where Θ denotes the noncommutative parameter [38] and its explicit value will be discussed later. Here the previous star-product is replaced with the ordinary product of matrices and the derivatives are represented as commutators. Note also that we abbreviate the higher derivative terms because we are interested in the limit $N \to \infty$.

In general it is difficult to know the explicit form of $f(T)$. Nevertheless we can determine its value before the tachyon condensation in such a way that it is consistent with the mass formula (2.10), namely

$$f(T=0) = \frac{p^2-1}{4N\alpha'}. \tag{3.4}$$

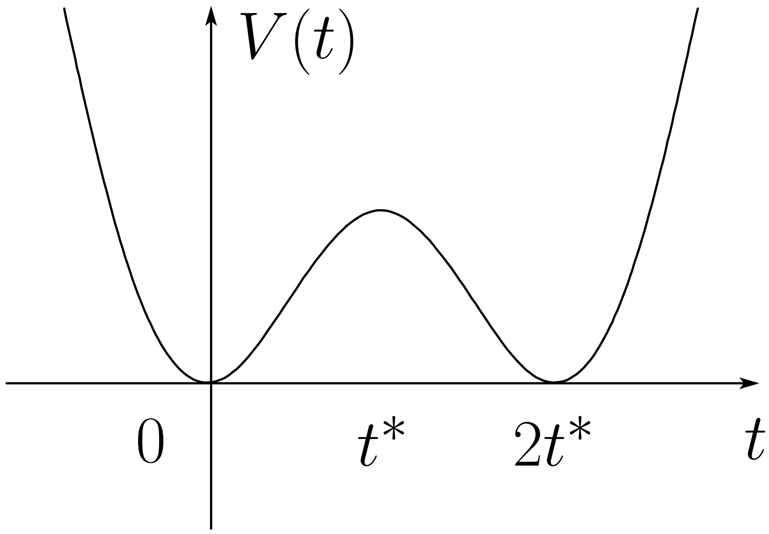

Figure 1: Tachyon potential on the non-BPS D-brane.

It is easy to see that in the limit $N \to \infty$ the kinetic term $\sim \frac{j(j+1)}{N\alpha'}$ is negligible and only the potential terms are relevant. Note that there is $U(p)$ symmetry $T \to gTg^{-1},\ \ g \in U(p)$ in this limit, which corresponds to the diffeomorphism of the fuzzy sphere. By using this symmetry we can diagonalize the matrix T.

Further we assume the double well potential of a non-BPS D-brane and shift the value of T as $T \to T + t^*$ for convenience. Here we define the local maximum of the function $V(t),\ t \in \mathbf{R}$ as t^* (see figure 1). This does not change the kinetic term and the potential term behaves as $V(T) = G_s T_{D2} - \frac{1}{2\alpha'}(T - t^*)^2 + \cdots$. As we will see, we can describe the tachyon condensation without knowing the detailed form of the potential.

Then we can describe the lightest excitation as $T = t^* \mathrm{diag}(0, \cdots, 0, 1, 0, \cdots, 0)$. Note that this matrix is a projection in the algebra of fuzzy sphere. We claim that the excitation is regarded as a "noncommutative soliton" [25] in a finite dimensional space; their applications to the tachyon condensation have been discussed in [26, 27, 28] recently. The mass of the original D2-brane is known to be p times as heavy as that of a D0-brane in the limit $N \to \infty$ from the analysis of the boundary state [23]. Note also that the original D2-brane corresponds to $T = t^* \mathrm{diag}(1, 1, \cdots, 1)$ in our model. Therefore we can see that the mass of the lightest excitation is the same as that of a D0-brane and identify the one with a D0-brane. Similarly the heavier excitations can be identified with multiple D0-branes.

Let us construct the soliton configuration on $\mathbf{S}^2$ in order to see more clearly that the excitation can be interpreted as a noncommutative soliton on the fuzzy sphere. On the

fuzzy sphere the spherical harmonics $Y_{j,m}$ is represented as the following $p \times p$ matrix [22]

$$
\begin{aligned}
(T_{j,m})_{m_1,m_2} &= (-1)^{\frac{p-1}{2}-m_1}\sqrt{2j+1}\begin{pmatrix} \frac{p-1}{2} & j & \frac{p-1}{2} \\ -m_1 & m & m_2 \end{pmatrix}, \\
\mathrm{Tr}[T_{j,m}T_{j',m'}] &= \delta_{j,j'}\delta_{m+m',0}(-1)^m,
\end{aligned} \tag{3.5}
$$

where we set $m_1, m_2 = -\frac{p-1}{2}, -\frac{p-3}{2}, \cdots, \frac{p-1}{2}$ and $(:::)$ denotes the $3j$-symbol. Then one can see that the matrix $T = t^*\mathrm{diag}(0^{\frac{p-1}{2}-a}, 1, 0^{\frac{p-1}{2}+a})$ corresponds to the following tachyon configuration

$$
T^{(a)}(\theta,\phi) = \frac{(-1)^{\frac{p-1}{2}-a}}{\sqrt{p}}t^*\sum_{l=0}^{p-1}(2l+1)\begin{pmatrix} \frac{p-1}{2} & l & \frac{p-1}{2} \\ -a & 0 & a \end{pmatrix}P_l(\cos\theta), \tag{3.6}
$$

where $P_l(\cos\theta)$ denote the Legendre polynomials. The overall normalization is determined by using that the trace corresponds to $\frac{1}{2\pi\Theta}\int d\phi \sin\theta d\theta$ and the explicit value of (3.10). Note that this is independent of ϕ and axially symmetric. We can construct the more general solution which is not symmetric by performing $U(p)$ rotation.

When $a = \frac{p-1}{2}$ the above can be written as

$$
T^{(\frac{p-1}{2})}(\theta,\phi) = \frac{1}{\sqrt{p}}t^*\sum_{l=0}^{p-1}(2l+1)\frac{(p-1)!}{\sqrt{(p+l)!(p-l-1)!}}P_l(\cos\theta). \tag{3.7}
$$

From this expression, it is not so difficult to see that in the region $p \gg 1, \theta \ll 1$ the configuration approximates[8] to the known noncommutative soliton

$$
T^{(\frac{p-1}{2})}(\theta) \sim 2t^* e^{-\frac{p}{2}\theta^2} \tag{3.8}
$$

in the flat space [25] (see figure 2). Similarly we can show

$$
T^{(-\frac{p-1}{2})}(\theta) \sim 2(-1)^{p-1}t^* e^{-\frac{p}{2}(\theta-\pi)^2}. \tag{3.9}
$$

From this one can read off the value of the noncommutative parameter for large p as

$$
\Theta = \frac{2}{p}\ . \tag{3.10}
$$

This value is the same as that obtained in [24], where the values of gauge flux and the B-field in the $N \to \infty$ limit were compared with the analysis given by Seiberg and Witten [38]. In this way we have verified that the excitation in this model can be regarded as a noncommutative soliton on the fuzzy sphere for finite and at least large p. Formally

[8]We confirmed this up to the normalization by exact calculations. We checked the overall normalization in the numerical method.

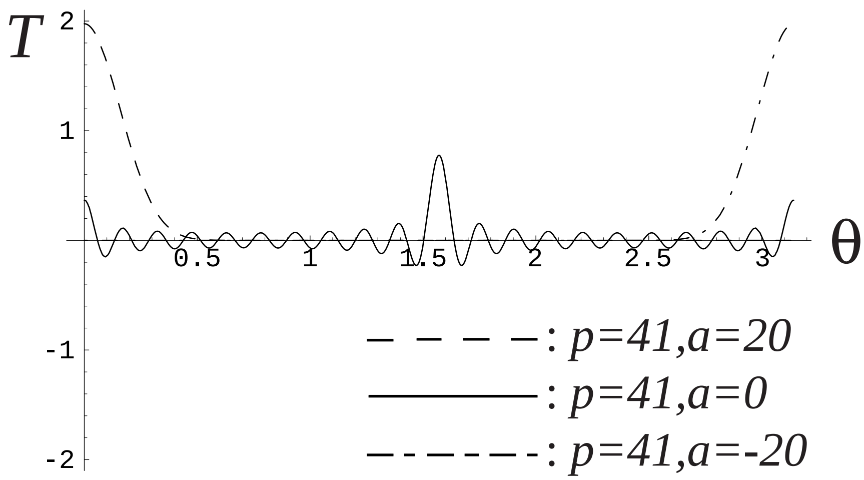

Figure 2: A plot of the equation (3.6). Here we set $t^* = 1$. The case $(p, a) = (41, 20)$ and $(p, a) = (41, -20)$ correspond to the solitons at the pole of the sphere. The case $(p, a) = (41, 0)$ corresponds to the one at the equator.

we can also consider the corresponding configuration (3.6) for small p, but the fuzziness $\delta x \sim \sqrt{\Theta}$ is comparable to the radius of the sphere and the name "soliton" does not seem to be appropriate.

It is natural to believe that the other cases ($a \neq \pm\frac{p-1}{2}$) are also treated as noncommutative solitons because they have the same energy. In paticular in the region $p \gg 1, \theta \ll 1, \frac{p-1}{2} - a \ll p$ we argue[9]

$$T^{(a)}(\theta) \sim 2(-1)^n t^* L_n(p\theta^2) e^{-\frac{p}{2}\theta^2}, \tag{3.11}$$

where L_n is Laguerre polynomial as discussed in [25] and we defined $n = \frac{p-1}{2} - a$. In other words, the solitons near the pole of sphere can be treated approximately as those in the flat space.

On the other hand, the solitons with $a \sim 0$ have a peak near the equator $\theta = \frac{\pi}{2}$ and is peculiar to the soliton on the sphere (see figure 2). Because we cannot construct a D1-brane in a conformal invariant way, it is natural to regard such a soliton as a quantum mechanically extended D0-brane. As we shall see below that such an object carries more energy than the others if N is finite. In any case the clear interpretation of such configurations in conformal field theory is certainly desirable.

In this way we have constructed the noncommutative soliton configurations of the tachyon field on the fuzzy sphere. Each of these has the mass of a D0-brane and obeys

[9]We checked this in the numerical method.

the following relation

$$\sum_{a=-\frac{p-1}{2}}^{\frac{p-1}{2}} T^{(a)}(\theta,\phi) = t^*. \tag{3.12}$$

If we consider an excitation $T^{(a_1)} + T^{(a_2)} + \cdots + T^{(a_k)}$ and set $a_i \neq a_j$ for $i \neq j$, then obviously we get k D0-branes. In the case of $k = p$ we get the original D2-brane.

We can also show that there are massless fields on the noncommutative soliton by closely following the discussion of [26]. As we mentioned above the low energy effective action of gauge fields on the fuzzy sphere was given by [20]. To include the coupling of tachyon field to gauge fields we can replace the derivatives with the covariant one. In the large p case, the action approaches the flat one but we should mention that now we deal with the finite dimensional case. Then it can be shown by following [26] that there are two massless scalars and $(p-1)$ W bosons on the lightest noncommutative soliton besides the tachyon[10]. When the soliton corresponds to k D0-branes, $U(p)$ symmetry breaks into $U(k) \times U(p-k)$ symmetry. It is interesting to investigate more detail but it is beyond the scope of this paper.

Next we discuss the effect of the kinetic term in (3.3). Such a term is important if N is finite and it breaks $U(p)$ symmetry into $SO(3)$. For $T^{(a)}$ we get

$$\mathrm{Tr}[\sum_{i=1,2,3} T^{(a)}[X^i,[X^i,T^{(a)}]\ ]\ ] = \frac{(t^*)^2}{p^2-1}(2p^2 - 8a^2 - 2). \tag{3.13}$$

This means that $a = -\frac{p-1}{2}$ and $a = \frac{p-1}{2}$ are the lowest energy configurations for finite N.

All of the above discussions hold for the D-brane in bosonic string. The exception is the existence of "massless" excitation observed in [26, 27, 28], which is peculiar to non-BPS D-branes. In our case this corresponds to $T = \mathrm{diag}(0,\cdots,0,2t^*,0,\cdots,0)$. Note that the tachyon potential of a non-BPS D-brane is double well and reaches its minimum at $T = 0, 2t^*$ (see figure 1). Though its physical interpretation is not clear also in our case, we can say that the interpretation proposed in [27] will not hold in our setup. In [27] the authors argued that the tensionless soliton corresponds to a BPS D1-brane winding around its core. However in the $SU(2)$ WZW model we cannot construct a D1-brane in any way[11].

As a final task let us consider D2-$\overline{\text{D2}}$ systems in the $N \to \infty$ limit. Here we denote the brane and the antibrane by the boundary state $|a\rangle$ and $|\bar{b}\rangle$ $(a \geq b)$, respectively. As in

[10]The slightly different approach has been quite recently proposed by [36] but the qualitative behavior does not seem to be changed.

[11] After we submitted the first version of this paper, there appeared an argument [39] that the "massless" excitation is gauge equivalent to the vacuum.

the previous case we can retain only the double well potential term and the total energy $E(T)$ is given by[12]

$$E(T) = \frac{1}{G_s}\mathrm{Tr}V(T) = M_{D0}\sum_{n=0}^{\infty} c_n \mathrm{Tr}\left[(TT^\dagger)^n + (T^\dagger T)^n\right], \tag{3.14}$$

where c_n $(c_0 = 1)$ are some constants and M_{D0} denotes the mass of D0-brane. The tachyon field T is complex $a \times b$ matrix and $T^\dagger$ denotes its hermitian conjugate. The overall normalization is determined by using the fact that $T = 0$ represents the original brane-antibrane system, namely $E(T=0) = M_{D2-\overline{D2}}$. Here $M_{D2-\overline{D2}}$ denotes the mass of the brane-antibrane system and is given by

$$M_{D2-\overline{D2}} = (a+b)M_{D0} \tag{3.15}$$

in the $N \to \infty$ limit. Let us define s^* such that $\hat{V}(s) = G_s M_{D0}\sum_{n=0}^{\infty} c_n s^n$ $(s \in \mathbf{R} > 0)$ take its minimum value at $s^*(> 0)$. The point $s = 0$ corresponds to the local maximum $\hat{V}(s = 0) = G_s M_{D0}$. Then the equation of motion is satisfied if

$$TT^\dagger T = s^* T, \quad T^\dagger T T^\dagger = s^* T^\dagger. \tag{3.16}$$

The same equation is obtained in [28, 40], where the noncommutative tachyon is discussed in a flat space. Note that the tachyon field in our case is a finite dimensional matrix.

If $a = b$, then the tachyon condensation can be treated as that on the non-BPS D-brane. The maximal condensation $T = \sqrt{s^*}e^{i\alpha}\mathrm{diag}(1,\cdots,1)$, where α is an arbitrary phase, means the decay to the vacuum and we get $\hat{V}(s^*) = 0$. Then it is easy to see that a tachyon configuration $T = \sqrt{s^*}e^{i\alpha}\mathrm{diag}(0^m, 1^{a-m})$ has its energy $E(T) = 2mM_{D0}$ and generates m D0-branes and m anti D0-branes.

Let us consider the more interesting case $a \neq b$. A series of solutions to (3.16) are given by

$$T^{(l)} = \sqrt{s^*}\begin{pmatrix} [1]_l & 0 \\ 0 & [0]_{b-l} \\ 0 & 0 \end{pmatrix} \quad (l = 0, 1, \cdots, b), \tag{3.17}$$

where $[a]_m$ denotes the $m \times m$ diagonal matrix $\mathrm{diag}(a, a, \cdots, a)$. General solutions are obtained by the further $U(a) \times U(b)$ gauge rotation. The energy of each solution is given by

$$E(T^{(l)}) = (a + b - 2l)M_{D0}. \tag{3.18}$$

[12]Unlike the previous case we do not shift the tachyon field T and the potential have the symmetry $V(T) = V(-T)$. This is only a matter of convention.

Furthermore the conservation of the RR-charge also guarantees that the total number of D0-branes remains $(a-b)$ in any process. Thus we get

$$\begin{array}{rcccl} \text{The number of D0-branes} & = & \dim\,(\text{Ker}\ T) & = & a-l\ , \\ \text{The number of anti D0-branes} & = & \dim\,(\text{Ker}\ T^\dagger) & = & b-l\ . \end{array} \tag{3.19}$$

If we take the difference between the above equations, we get

$$\text{Index}(T) = a-b = \int_{\mathbf{S}^2} \frac{F^{(a)}-F^{(b)}}{2\pi}, \tag{3.20}$$

where $F^{(a)}$ and $F^{(b)}$ denote the gauge fluxes on the world-volume of the brane and the antibrane, respectively. Their explicit values are given by (2.5).

The relation between the noncommutative tachyon in the brane-antibrane system and K-theory of noncommutative algebras has been discussed quite recently in [28, 41]. Our result offers a special example of this, where the (finite dimensional) matrix algebra is involved. We argue that the relation (3.20) is regarded as a generalized index theorem of noncommutative algebras (see for example [42, 43]). The left-hand side of (3.20) is obtained from the vertex operator analysis and the right-hand side represents that the D2-brane carries the K-theory charge of D0-brane. Therefore we can see that the requirement that this relation is consistent with the index theorem ensures the bound state interpretation. In order to see the relation between the Index(T) and K-theory charge more clearly, it would be interesting to investigate the tachyon condensation in the other rational conformal field theories.

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